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Resolution of accessible singularities of a third order differential equation

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0. Introduction

In this paper, we study some third order differential equations and their space of initial conditions. The most important property of the Painlvé equation is the so called Painlevé property, namely, every solution of each Painlevé equation has neither movable branch points nor movable essential singularities. Let E be the set of the fixed singular points of Painlevé equation and let $B = \mathbb{P}^1 - E$. The Painlevé property is stated as: any local solutions $x(t)$ of Painlevé equation (they are determined by an arbitary initial condition $x(t_0) = x_0, (dx/dt)(t_0) = x_1$ with $t_0 \in B$ and with a certain condition on x_0) can be meromorphically continued along any curve in B. If $(x(t), y(t))$ is a solution which is determined by an arbitary initial condition $x(t_0) = x_0 \in \mathbb{C}, y(t_0) = y_0 \in \mathbb{C}$ with $t_0 \in B$, then both $(x(t), y(t))$ can be meromorphically continued along any curve in B with a starting point t_0. K. Okamoto constructed a minimal fiber space in which every solution stays. The fiber space $\mathcal{S}$ is constructed as follows. Firstly, we take the Hirzebruch surface T as a minimal compactification of $\mathbb{C}^2$. Secondly, we apply a finite number of quadric transformations to the space T by carefully observing the forms of the Hamiltonian systems in new variables transformed from the original system, and obtain a compact space $\tilde{\mathcal{S}}$. Lastly, we remove, from $\tilde{\mathcal{S}}$, a finite number of divisors which consist of vertical leaves and singular points of the foliation, and obtain $\mathcal{S}$. (Here a vertical leaf is a leaf which is completely included in $\tilde{\mathcal{S}}$.) We can obtain the space of initial conditions by adding spaces of codimension 1 to the original space. But in the case of third order differential equations, we need to add codimension 2 spaces to the original space in addition to codimension 1 spaces. And when we resolve the accessible singularities of the third order differential equation or the Noumi-Yamada equation, we blow up them and need to blow down. There is the reason why the codimension of the meromorphic solutions extending through these accessible singular points may be less than equal to two. Moreover we need to resolve for a pair of these accessible singularities. In this paper, we shall explain the birational transformations which gives the resolution of these accessible singularities.

1. Statement of main results

Let us consider the system of Noumi-Yamada of type $A_4{}^{(1)}$ in the following.

$$\begin{cases} f_0' = f_0(f_1 - f_2 + f_3 - f_4) + \alpha_0 \\ f_1' = f_1(f_2 - f_3 + f_4 - f_0) + \alpha_1 \\ f_2' = f_2(f_3 - f_4 + f_0 - f_1) + \alpha_2 \\ f_3' = f_3(f_4 - f_0 + f_1 - f_2) + \alpha_3 \\ f_4' = f_4(f_0 - f_1 + f_2 - f_3) + \alpha_4, \end{cases} \tag{1}$$

where $f_0' + f_1' + f_2' + f_3' + f_4' = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1$, $f_0 + f_1 + f_2 + f_3 + f_4 = t$.

Setting $f_1 = 0, \alpha_1 = 0$ and $x := f_2, y := f_3, z := f_4$, the system is reduced to the following system

$$\begin{cases} \dfrac{dx}{dt} = x(t - x - 2z) + \alpha_2 \\ \dfrac{dy}{dt} = y(-t + y + 2z) + \alpha_3 \\ \dfrac{dz}{dt} = z(t - 2y - z) + \alpha_4. \end{cases} \tag{2}$$

The last two equations of this system (2) is equivalent to the fourth Painlevé equations with unkown variables (y, z). For each solution $z(t)$ of the Painlevé equation, the first equation of the system (2) gives a Riccati equation with unknown variable x. In order to consider the space of initial conditions for the system (2), let us the compactification $\mathbb{P}^3$ of $\mathbb{C}^3$ with a natural embedding

$$\begin{array}{ccc} \mathbb{C}^3 & \hookrightarrow & \mathbb{P}^3 \\ (x, y, z) & \mapsto & [z_0, z_1, z_2, z_3] = [1 : x : y : z]. \end{array}$$

Moreover we denote by $H = \{z_0 = 0\} \simeq \mathbb{P}^2 \subset \mathbb{P}^3$. Fixing parameter α_i, consider the product $\mathbb{P}^3 \times B$ and extend the regular vector field

$$v = \frac{\partial}{\partial t} + \{x(t - x - 2z) + \alpha_2\}\frac{\partial}{\partial x} + \{y(-t + y + 2z + \alpha_3\}\frac{\partial}{\partial y} + \{z(t - 2y - z) + \alpha_4\}\frac{\partial}{\partial z}$$

on $\mathbb{C}^3 \times B$ to a rational vector field $\tilde{v}$ on $\mathbb{P}^3 \times B$. Then it is easy to see that this rational vector field $\tilde{v}$ has seven accessible singularities on the boundary divisor $H \times \{t\} \subset \mathbb{P}^3 \times \{t\}$ for each $t \in B$.

The purpose of this paper is to give an explicit resolutions of these accesible singularities and to obtain a nice space of initial conditions on which the rational vector field $\tilde{v}$ has no accessible singularity. The following is our main result.

Theorem 1.1. *After a series of expricit blowings-up and blowings-down of $\mathbb{P}^3 \times B$, we obtain a smooth projective family of 3-fold $\pi : \mathcal{X} \longrightarrow B$ and a birational morphism $\varphi : \mathcal{X} \cdots \longrightarrow \mathbb{P}^3 \times B$ which make the following diagram commutative*

$$\begin{array}{ccc} \mathcal{X} & \overset{\varphi}{\cdots\rightarrow} & \mathbb{P}^3 \times B \\ \pi \downarrow & & \downarrow \\ B & = & B, \end{array}$$

and satisfies the following conditions,

1. *There exists a flat divisor $\mathcal{D}$ over B such that the vector field $\tilde{v}$ associated to the third order differential equation becomes a regular vector field on $\mathcal{X} \setminus \mathcal{D}$ and*
$$\tilde{v} \in H^0(\mathcal{X}, \Theta_{\mathcal{X}}(-\log \mathcal{D})(\mathcal{D})).$$
2. *The vector field $\tilde{v}$ has no accessible singularity along the boundary divisor $\mathcal{D}$.*
3. *The inverse birational map φ^{-1} restricted to the open set $\mathbb{C}^3 \times B = (\mathbb{P}^3 \setminus H) \times B$ is an isomorphism onto its image $\varphi^{-1} : \mathbb{C}^3 \times B \xrightarrow{\simeq} \varphi^{-1}(\mathbb{C}^3 \times B) \subset \mathcal{X} \setminus \mathcal{D}$.*
4. *The zariski open subset $\mathcal{X} \setminus \mathcal{D}$ is a space of initial conditions for the vector field $\tilde{v}$ and $\mathcal{X} \setminus \mathcal{D}$ can be covered by affine charts.*

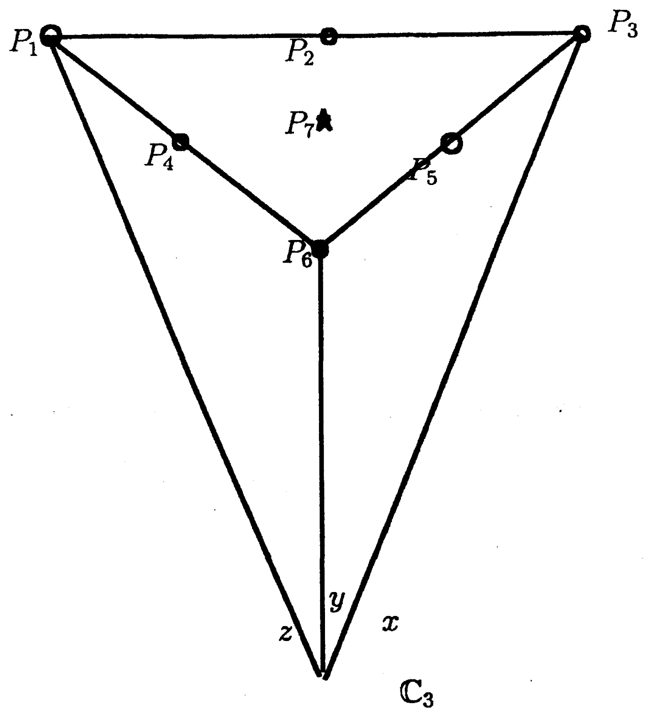

FIGURE 1

5. *Each fiber* $(\mathcal{X} \setminus \mathcal{D})_t$ *of the morphism* $\pi : \mathcal{X} \setminus \mathcal{D} \longrightarrow B$ *has a stratification*

$$(\mathcal{X} \setminus \mathcal{D})_t = \mathbb{C}^3 \cup \mathbb{C}^2 \cup \mathbb{C}^2 \cup \mathbb{C}^2 \cup \mathbb{C}^2 \cup \mathbb{C}^1 \cup \mathbb{C}^1 \cup \mathbb{C}^1$$

where $\mathbb{C}^3$ *is the original affine open subset* $\mathbb{P}^3 \setminus H$.

2. ACCESSIBLE SINGULARITIES

There exist 7 accessible singularities of this equation on the boundary divisor of $\mathbb{P}^3$. They are listed as follows. (See Figure 1).

Singular points	P_1	P_2	P_3	P_4
Coordinates $[z_0, z_1, z_2, z_3]$	$[0,0,0,1]$	$[0,-1,0,1]$	$[0,1,0,0]$	$[0,0,-1,0]$
Type (dim. of sol.)	$\circ$ (dim. 1)	$\bullet$ (dim. 2)	$\bullet$ (dim. 2)	$\bullet$ (dim. 2)
Singular points	P_5	P_6	P_7	
Coordinates $[z_0, z_1, z_2, z_3]$	$[0,-1,1,0]$	$[0,0,1,0]$	$[0,-3,-1,1]$	
Type (dim. of sol.)	$\circ$ (dim. 1)	$\bullet$ (dim. 1)	$\star$ (dim. 2)	

Here, we remark that there are 3 types of accessible singularities. We denote by $\bullet$ an accessible singularity into which two dimensional meromorphic local solutions flow, by $\circ$ and $\star$ accessible singularities into which one dimensional meromorphic local solutions flow. The difference of $\circ$ and $\star$ can be distinguished by the process of the resolutions.

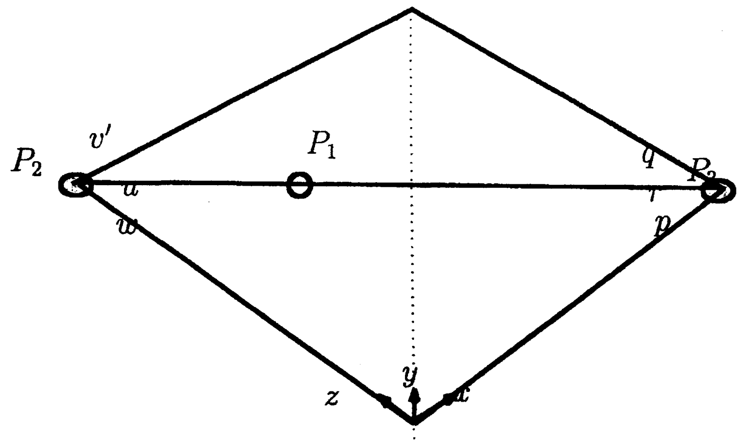

FIGURE 2

3. RESOLUTION OF THE ACCESSIBLE SINGULARITIES

3.1. Resolution of the accessible singularities $P_1 : (u, \tilde{v}, w) = (0,1,0), P_2 : (u, \tilde{v}, w) = (0,0,0), P_3 : (p, q, r) = (0,0,0)$, where $\tilde{v} = v + 1$.

Around P_2: $(u, \tilde{v}, w) = (0,0,0)$, the equation is given by the following.

$$\begin{cases} \dfrac{du}{dt} = \dfrac{u - u^2 - \tilde{v}(2 - 2u)}{w} + \alpha_4 w + \alpha_2 w - \alpha_4 u w \\ \dfrac{d\tilde{v}}{dt} = -2t\tilde{v} + \dfrac{3\tilde{v}(1 + \tilde{v})}{w} + \alpha_3 w - \alpha_4 \tilde{v} w \\ \dfrac{dw}{dt} = -\alpha_4 w^2 - tw + 2\tilde{v} + 1 \end{cases} \tag{3}$$

Around P_1: $(u, v, w) = (0,0,0)$, the equation is given by the following.

$$\begin{cases} \dfrac{du}{dt} = \dfrac{-u - u^2 + 2uv}{w} + \alpha_2 w - \alpha_4 u w \\ \dfrac{dv}{dt} = -2tv + \dfrac{3v + 3v^2}{w} + \alpha_3 w - \alpha_4 v w \\ \dfrac{dw}{dt} = 1 + 2v - tw - \alpha_4 w^2 \end{cases} \tag{4}$$

Around P_3: $(p, q, r) = (0,0,0)$, the equation is given by the following.

$$\begin{cases} \dfrac{dp}{dt} = -\alpha_2 p^2 - tp + 2r + 1 \\ \dfrac{dq}{dt} = \dfrac{q(-2tp + q + 4r + 1)}{p} + \alpha_3 p - \alpha_2 p q \\ \dfrac{dr}{dt} = \dfrac{r(1 + r - 2q)}{p} - \alpha_2 p r + \alpha_4 p \end{cases} \tag{5}$$

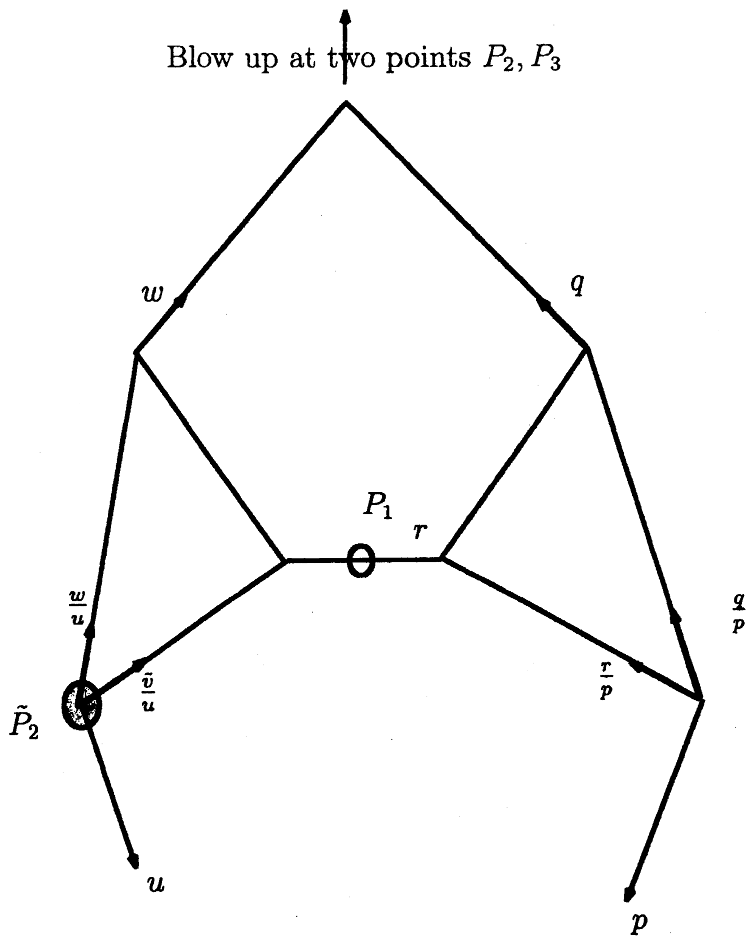

FIGURE 3

Around the point P_1: $(u,v,w)=(0,0,0)$, the equation is given by the following.

$$\begin{cases} \dfrac{du}{dt} = \dfrac{-u-u^2+2uv}{w} + \alpha_2 w - \alpha_4 uw \\ \dfrac{dv}{dt} = -2tv + \dfrac{3v+3v^2}{w} + \alpha_3 w - \alpha_4 vw \\ \dfrac{dw}{dt} = 1 + 2v - tw - \alpha_4 w^2 \end{cases} \tag{6}$$

$$\begin{cases} p_1 &= p \\ q_1 &= \frac{q}{p} \\ r_1 &= \frac{r}{p} \end{cases} \tag{7}$$

Around $(p_1,q_1,r_1)=(0,0,0)$, the equation is given by the following.

$$\begin{cases} \dfrac{dp_1}{dt} = -tp_1 + 1 + 2p_1 r_1 - \alpha_2 (p_1)^2 \\ \dfrac{dq_1}{dt} = (q_1)^2 - tq_1 + 2q_1 r_1 + \alpha_3 \\ \dfrac{dr_1}{dt} = tr_1 - (r_1)^2 - 2q_1 r_1 + \alpha_4 \end{cases} \tag{8}$$

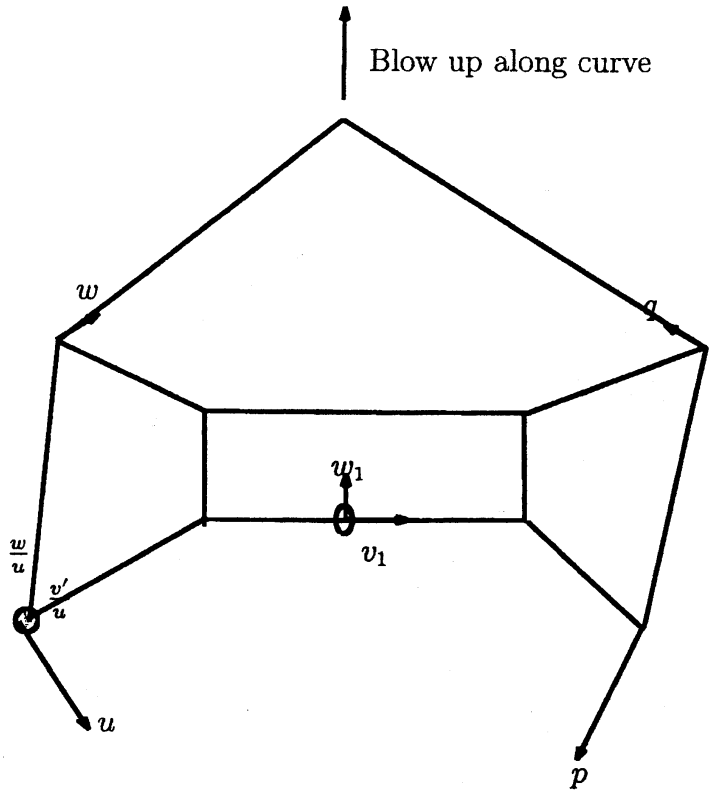

FIGURE 4

We blow up along curve $\mathbb{P}^1$, then we obtain a $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$ and denote it's suface F . The curve $C = [(u_1, v_1, w_1)|u_1 = w_1 = 0]$ in F is (-1) curve when we restrict to F. So we can blow down the suface F along a curve and we can obtain a smooth rational variaty.

$$\begin{cases} u_1 &= u \\ v_1 &= \frac{v}{w} \\ w_1 &= w \end{cases} \tag{9}$$

$$\begin{cases} \dfrac{du_1}{dt} = 2u_1v_1 + \dfrac{-u_1 - (u_1)^2}{w_1} + \alpha_2 w_1 - \alpha_4 u_1 w_1 \\ \dfrac{dv_1}{dt} = -tv_1 + (v_1)^2 + \dfrac{2v_1}{w_1} + \alpha_3 \\ \dfrac{dw_1}{dt} = 1 - tw_1 + 2v_1w_1 - \alpha_4(w_1)^2 \end{cases} \tag{10}$$

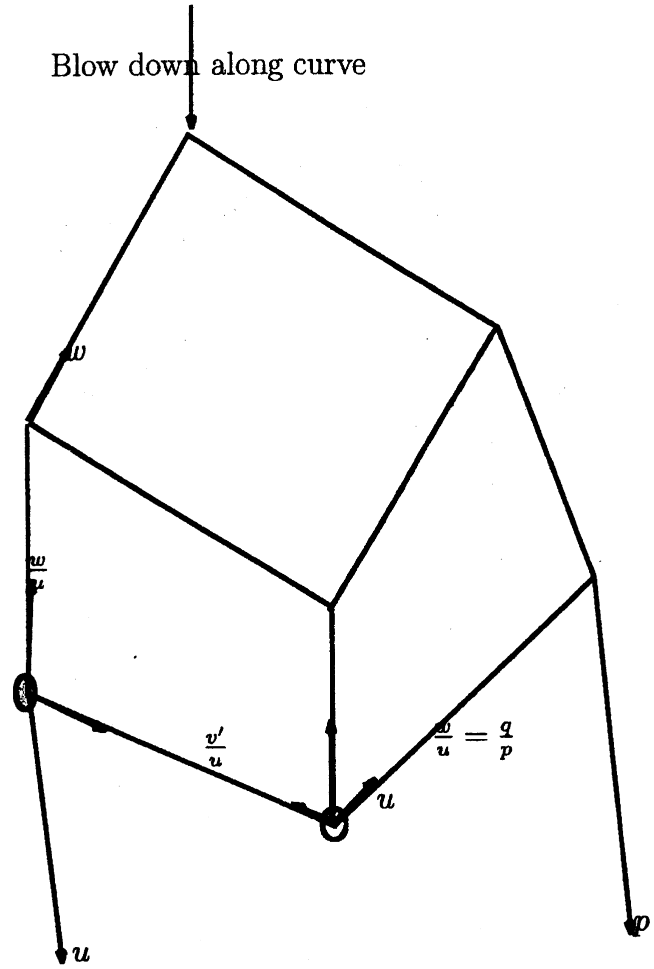

FIGURE 5

Let us blow up at two points P_2, P_3 and blow up along a curve. Then we obtain $\mathbb{P}^1 \times \mathbb{P}^1$ and we can blow down along curve $\mathbb{P}^1$. As a results, we obtain a smooth rational variety. In this way, we need to resolve a pair of some accessible singularities when there are accessible singularities whose codimension are 2.

$$\begin{cases} u_2 &= \frac{w_1}{u_1+1} \\ v_2 &= v_1 \\ w_2 &= w_1 \end{cases} \tag{11}$$

$$\begin{cases} \dfrac{du_2}{dt} = 1 - tu_2 + \dfrac{2(u_2)^2 v_2}{w_2} - \alpha_2 (w_2)^2 - \alpha_4 (w_2)^2 \\ \dfrac{dv_2}{dt} = -tv_2 + (v_2)^2 + \dfrac{2v_2}{w_2} + \alpha_3 \\ \dfrac{dw_2}{dt} = 1 - tw_2 + 2v_2 w_2 - \alpha_4 (w_2)^2 \end{cases} \tag{12}$$

We blow up along curve $\mathbb{P}^1$ two times, then we can resolve the accessible singularities P_1, P_2. In the new coordinate, the right hand side of the equations is the form of polynomials.

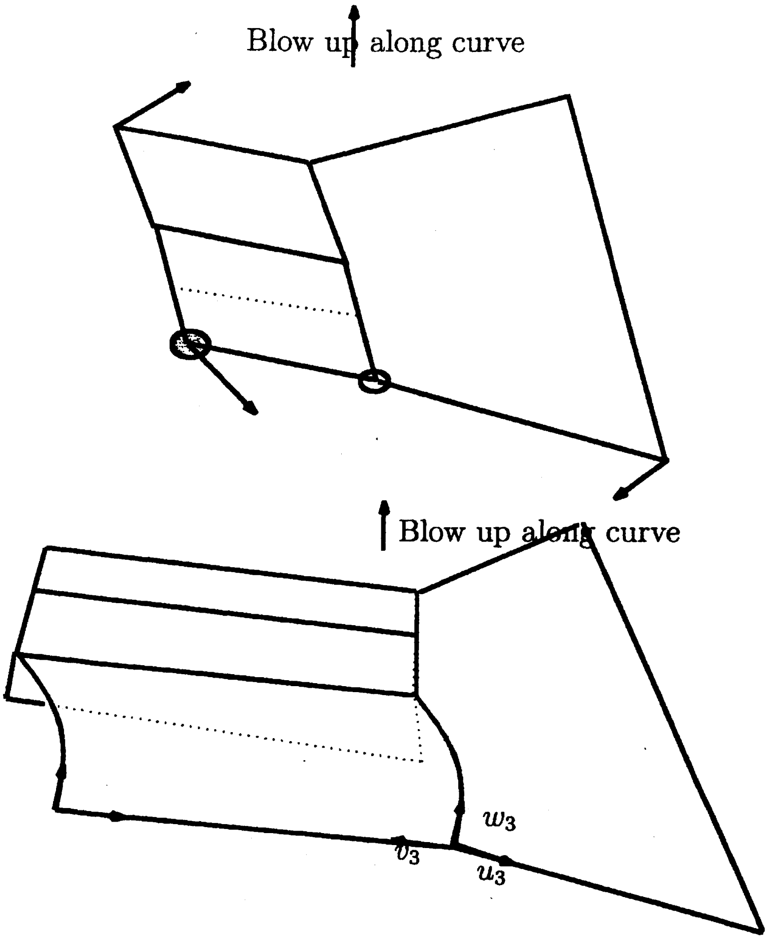

FIGURE 6

$$\begin{cases} u_3 &= u_2 \\ v_3 &= \frac{v_2+\alpha_3 w_2}{(w_2)^2} \\ w_3 &= w_2 \end{cases} \tag{13}$$

Around $(u_3, v_3, w_3) = (0,0,0)$, the equation is given by the following.

$$\begin{cases} \dfrac{du_3}{dt} = 1 - tu_3 + 2(u_3)^2 v_3 w_3 - \alpha_2 (u_3)^2 - 2\alpha_3 (u_3)^2 - \alpha_4 (u_3)^2 \\ \dfrac{dv_3}{dt} = tv_3 - 3(v_3)^2 (w_3)^2 + 4\alpha_3 v_3 w_3 - (\alpha_3)^2 + 2\alpha_4 v_3 w_3 - \alpha_3 \alpha_4 \\ \dfrac{dw_3}{dt} = -\alpha_4 (w_3)^2 - tw_3 + 2(w_3)^2 (v_3 w_3 - \alpha_3) + 1 \end{cases} \tag{14}$$

3.2 Resolution of the accessible singularities $P_5 : (l,m,n) = (0,-1,0), P_6 : (l,m,n) = (0,0,0)$

We can resolve the accessible singularities P_5, P_6 by the same way of 3.1.

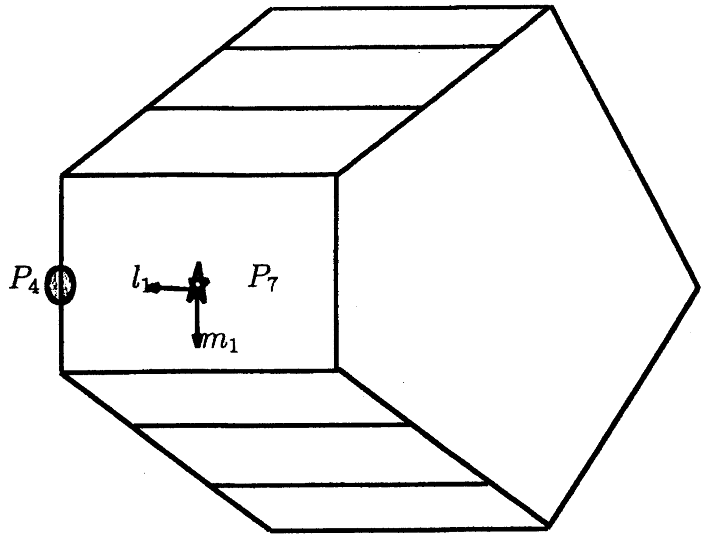

FIGURE 7

3.3 Resolution of the accessible singularities P_4 : $(u,v,w) = (0,-1,0), P_7$: $(u,v,w) = (-3,-1,0)$

$$\begin{cases} l_1 &= \frac{x}{z}+3 \\ m_1 &= \frac{y}{z}+1 \\ n_1 &= \frac{1}{z} \end{cases} \tag{15}$$

In the coordinate, the equation is given by the following.

$$\begin{cases} \dfrac{dl_1}{dt} = \dfrac{3l_1-(l_1)^2-6m_1+2l_1m_1}{n_1}+n_1\alpha_2+3n_1\alpha_4-l_1n_1\alpha 4 \\ \dfrac{dm_1}{dt} = \dfrac{-3m_1+3(m_1)^2}{n_1}+2t-2tm_1+n_1\alpha_3+n_1\alpha_4-m_1n_1\alpha_4 \\ \dfrac{dn_1}{dt} = -1+2m_1-tn_1-\alpha_4(n_1)^2 \end{cases} \tag{16}$$

$$\begin{cases} L_1 &= \frac{x}{z} \\ M_1 &= \frac{y}{z}+1 \\ N_1 &= \frac{1}{z} \end{cases} \tag{17}$$

In the coordinate, the equation is given by the following.

$$\begin{cases} \dfrac{dL_1}{dt} = \dfrac{-3L_1-(L_1)^2+2L_1M_1}{N_1}+\alpha_2N_1-\alpha_4L_1N_1 \\ \dfrac{dM_1}{dt} = 2t-2tM_1+\dfrac{-3M_1+3(M_1)^2}{N_1}+\alpha_3N_1+\alpha_4N_1-\alpha_4M_1N_1 \\ \dfrac{dN_1}{dt} = -1+2M_1-tN_1-\alpha_4(N_1)^2 \end{cases} \tag{18}$$

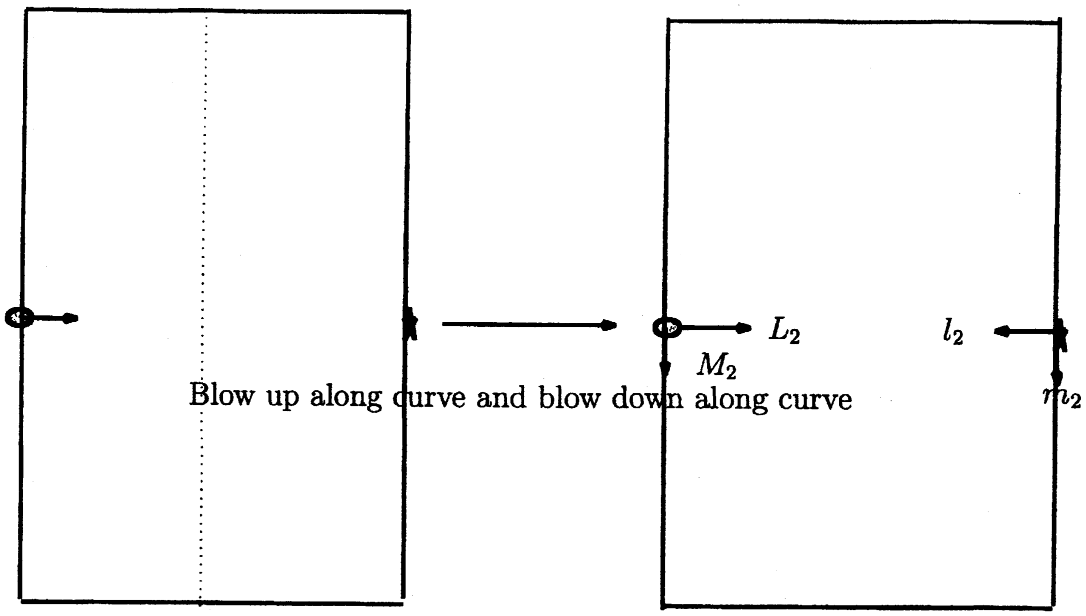

FIGURE 8

$$\begin{cases} l_2 & = \frac{n_1}{l_1-3} \\ m_2 & = m_1 \\ n_2 & = n_1 \end{cases} \tag{19}$$

In the coordinate, the equation is given by the following.

$$\begin{cases} \dfrac{dl_2}{dt} = 1 - tl_2 + \dfrac{2l_2}{n_2} - \alpha_2(l_2)^2 \\ \dfrac{dm_2}{dt} = 2t - 2tm_2 - \dfrac{3m_2 - 3(m_2)^2}{n_2} + \alpha_3 n_2 + \alpha_4 n_2 - \alpha_4 m_2 n_2 \\ \dfrac{dn_2}{dt} = -1 + 2m_2 - tn_2 - \alpha_4(n_2)^2 \end{cases} \tag{20}$$

$$\begin{cases} L_2 & = \frac{L_1}{N_1} \\ M_2 & = M_1 \\ N_2 & = N_1 \end{cases} \tag{21}$$

In the coordinate, the equation is given by the following.

$$\begin{cases} \dfrac{dL_2}{dt} = tL_2 - (L_2)^2 - \dfrac{2L_2}{N_2} + \alpha_2 \\ \dfrac{dM_2}{dt} = 2t - 2tM_2 + \dfrac{-3M_2 + 3(M_2)^2}{N_2} + \alpha_3 N_2 + \alpha_4 N_2 - \alpha_4 M_2 N_2 \\ \dfrac{dN_2}{dt} = -1 + 2M_2 - tN_2 - \alpha_4(N_2)^2 \end{cases} \tag{22}$$

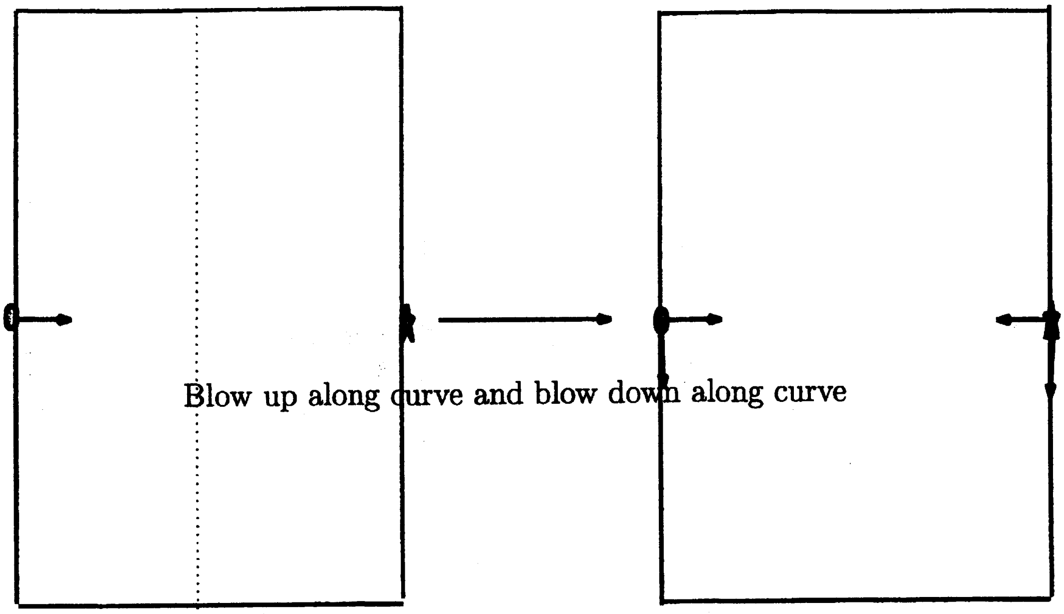

FIGURE 9

$$\begin{cases} l_3 &= l_2 n_2 \\ m_3 &= m_2 \\ n_3 &= n_2 \end{cases} \tag{23}$$

In the coordinate, the equation is given by the following.

$$\begin{cases} \dfrac{dl_3}{dt} = -2tl_3 + \dfrac{2l_3 m_3 - \alpha_2 (l_3)^2}{n_3} + n_3 - \alpha_4 l_3 n_3 \\ \dfrac{dm_3}{dt} = 2t - 2tm_3 - \dfrac{3m_3 - 3(m_3)^2}{n_3} + \alpha_3 n_3 + \alpha_4 n_3 - \alpha_4 m_3 n_3 \\ \dfrac{dn_3}{dt} = -1 + 2m_3 - tn_3 - \alpha_4 (n_3)^2 \end{cases} \tag{24}$$

$$\begin{cases} L_3 &= \frac{L_2}{N_2} \\ M_3 &= M_2 \\ N_3 &= N_2 \end{cases} \tag{25}$$

In the coordinate, the equation is given by the following.

$$\begin{cases} \dfrac{dL_3}{dt} = 2tL_3 + \dfrac{-L_3 - 2L_3 M_3 + \alpha_2}{N_3} - (L_3)^2 N_3 + \alpha_4 L_3 N_3 \\ \dfrac{dM_3}{dt} = 2t - 2tM_3 + \dfrac{-3M_3 + 3(M_3)^2}{N_3} + \alpha_3 N_3 + \alpha_4 N_3 - \alpha_4 M_3 N_3 \\ \dfrac{dN_3}{dt} = -1 + 2M_3 - tN_3 - \alpha_4 (N_3)^2 \end{cases} \tag{26}$$

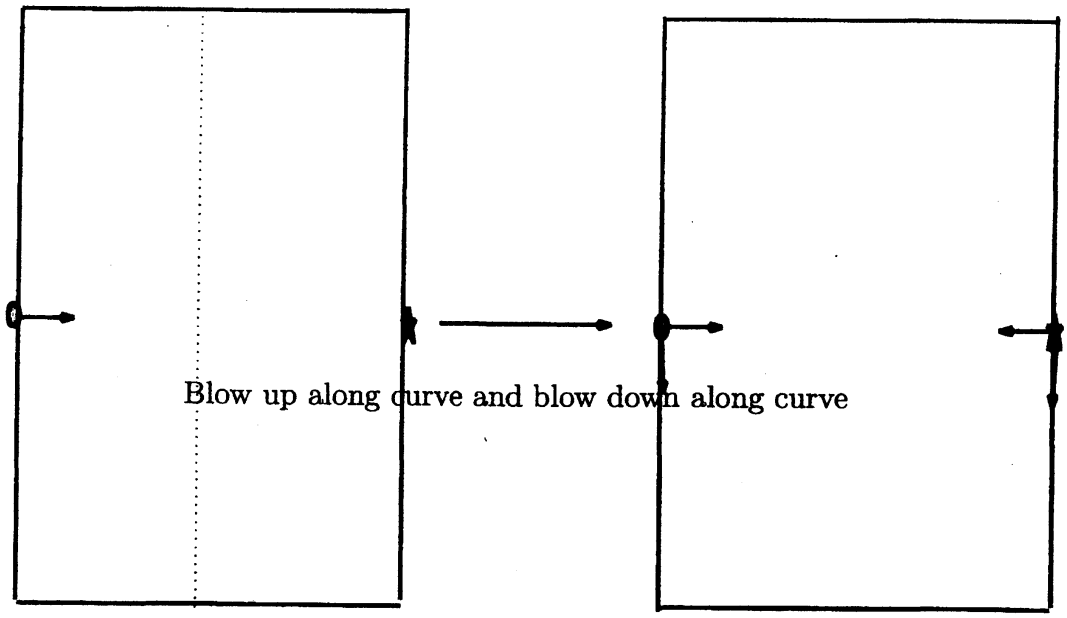

FIGURE 10

$$\begin{cases} l_4 &= \frac{l_2 n_2}{1-\alpha_2 l_3} \\ m_4 &= m_3 \\ n_4 &= n_3 \end{cases} \tag{27}$$

In the coordinate, the equation is given by the following.

$$\begin{cases} \dfrac{dl_4}{dt} = -3tl_4 + \dfrac{4l_4 m_4 n_4 + 2\alpha_2 (l_4)^2 m_4 - 2t\alpha_2 (l_4)^2 n_4}{(n_4)^2} + 2\alpha_2 l_4 n_4 \\ \qquad + (\alpha_2)^2 (l_4)^2 - \alpha_4 l_4 n_4 - \alpha_2 \alpha_4 (l_4)^2 \\ \dfrac{dm_4}{dt} = 2t - 2tm_4 - \dfrac{3m_4 - 3(m_4)^2}{n_4} + \alpha_3 n_4 + \alpha_4 n_4 - \alpha_4 m_4 n_4 \\ \dfrac{dn_4}{dt} = -1 + 2m_4 - tn_4 - \alpha_4 (n_4)^2 \end{cases} \tag{28}$$

$$\begin{cases} L_4 &= \frac{L_3 - \alpha_2}{N_3} \\ M_4 &= M_3 \\ N_4 &= N_3 \end{cases} \tag{29}$$

In the coordinate, the equation is given by the following.

$$\begin{cases} \dfrac{dL_4}{dt} = 3tN_4 + \dfrac{-4L_4 M_4 N_4 - 2\alpha_2 M_4 N_4 + 2t\alpha_2 N_4}{(N_4)^2} \\ \qquad - (L_4 M_4)^2 - 2\alpha_2 L_4 N_4 - (\alpha_2)^2 + 2\alpha_4 L_4 n_4 + \alpha_2 \alpha_4 \\ \dfrac{dM_4}{dt} = 2t - 2tM_4 + \dfrac{-3M_4 + 3(M_4)^2}{N_4} + \alpha_3 N_4 + \alpha_4 N_4 - \alpha_4 M_4 N_4 \\ \dfrac{dN_4}{dt} = -1 + 2M_4 - tN_4 - \alpha_4 (N_4)^2 \end{cases} \tag{30}$$

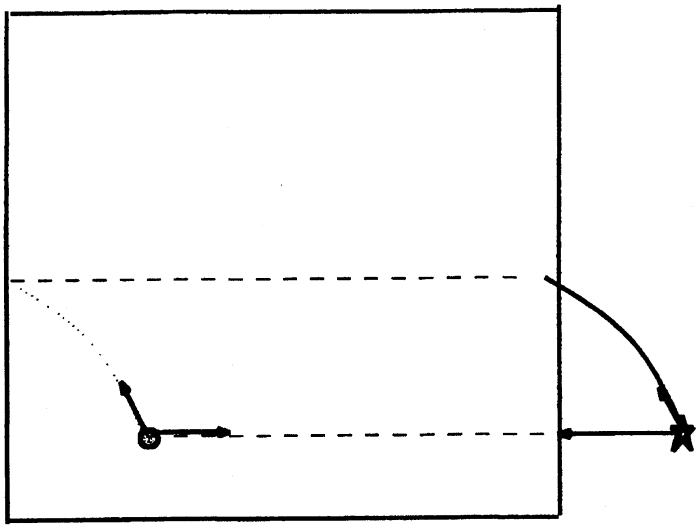

FIGURE 11

$$\begin{cases} l_5 &= l_4 \\ m_5 &= \frac{m_4}{n_4} \\ n_5 &= n_4 \end{cases} \tag{31}$$

In the coordinate, the equation is given by the following.

$$\begin{cases} \dfrac{dl_5}{dt} = -3tl_5 + 4l_5m_5 + \dfrac{2\alpha_2(l_5)^2m_5 - 2t\alpha_2(l_5)^2}{n_5} + 2\alpha_2 l_5 n_5 + (\alpha_2)^2(l_5)^2 \\ \qquad - 2\alpha_4 l_5 n_5 - \alpha_2\alpha_4(l_5)^2 \\ \dfrac{dm_5}{dt} = -tm_5 + (m_5)^2 + \dfrac{2t - 2m_5}{n_5} + \alpha_3 + \alpha_4 \\ \dfrac{dn_5}{dt} = -1 + 2m_5n_5 - tn_5 - \alpha_4(n_5)^2 \end{cases} \tag{32}$$

$$\begin{cases} L_5 &= L_4 \\ M_5 &= \frac{M_4}{N_4} \\ N_5 &= N_4 \end{cases} \tag{33}$$

In the coordinate, the equation is given by the following.

$$\begin{cases} \dfrac{dL_5}{dt} = 3tL_5 - 4L_5M_5 + \dfrac{-2\alpha_2M_5 + 2t\alpha_2}{N_5} \\ \qquad - (L_5M_5)^2 - 2\alpha_2L_5N_5 - (\alpha_2)^2 + 2\alpha_4L_5N_5 + \alpha_2\alpha_4 \\ \dfrac{dM_5}{dt} = -tM_5 + (M_5)^2 + \dfrac{2t - 2M_5}{N_5} + \alpha_3 + \alpha_4 \\ \dfrac{dN_5}{dt} = -1 + 2M_5N_5 - tN_5 - \alpha_4(N_5)^2 \end{cases} \tag{34}$$

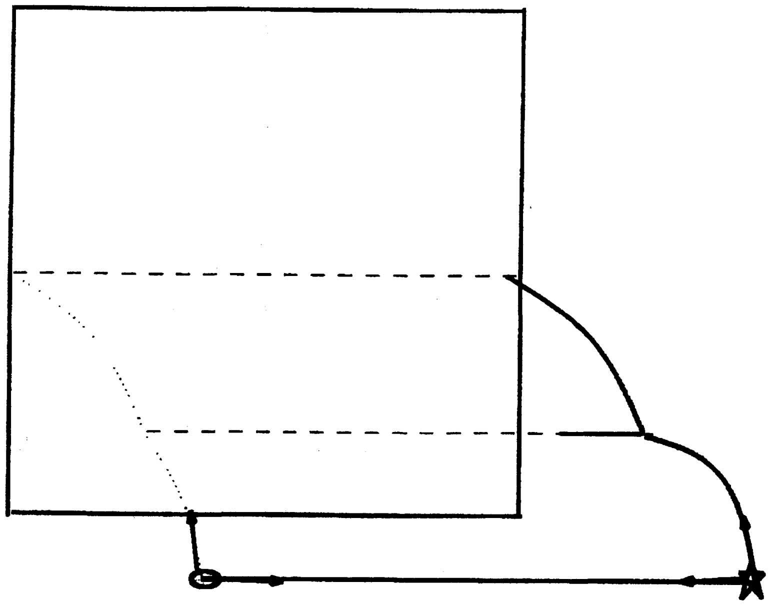

FIGURE 12

$$\begin{cases} l_6 &= l_5 \\ m_6 &= \frac{m_5 - t}{n_5} \\ n_6 &= n_5 \end{cases} \tag{35}$$

In the coordinate, the equation is given by the following.

$$\begin{cases} \dfrac{dl_6}{dt} = tl_6 + 4l_6m_6n_6 + (n_6)^2 + 2\alpha_2(l_6)^2m_6 + 2\alpha_2l_6n_6 + (\alpha_2)^2(l_6)^2 \\ \qquad - 2\alpha_4l_6n_6 - \alpha_2\alpha_4(l_6)^2 \\ \dfrac{dm_6}{dt} = \dfrac{-m_6 - 1 + \alpha_3 + \alpha_4}{n_6} - (m_6)^2n_6 + \alpha_4m_6n_6 \\ \dfrac{dn_6}{dt} = -1 + 2m_6(n_6)^2 + tn_6 - \alpha_4(n_6)^2 \end{cases} \tag{36}$$

$$\begin{cases} L_6 &= L_5 \\ M_6 &= \frac{M_5 - t}{N_5} \\ N_6 &= N_5 \end{cases} \tag{37}$$

In the coordinate, the equation is given by the following.

$$\begin{cases} \dfrac{dL_6}{dt} = -tL_6 - 4L_6M_6N_6 - (L_6M_6)^2 - 2\alpha_2L_6 - \\ \qquad 2\alpha_2L_6N_6 - (\alpha_2)^2 + 2\alpha_4L_6N_6 + \alpha_2\alpha_4 \\ \dfrac{dM_6}{dt} = -(M_6)^2N_6 + \dfrac{-1 - M_6 + \alpha_3 + \alpha_4}{N_6} + \alpha_4M_6N_6 \\ \dfrac{dN_6}{dt} = -1 + 2M_6N_6 + tN_6 - \alpha_4(N_6)^2 \end{cases} \tag{38}$$

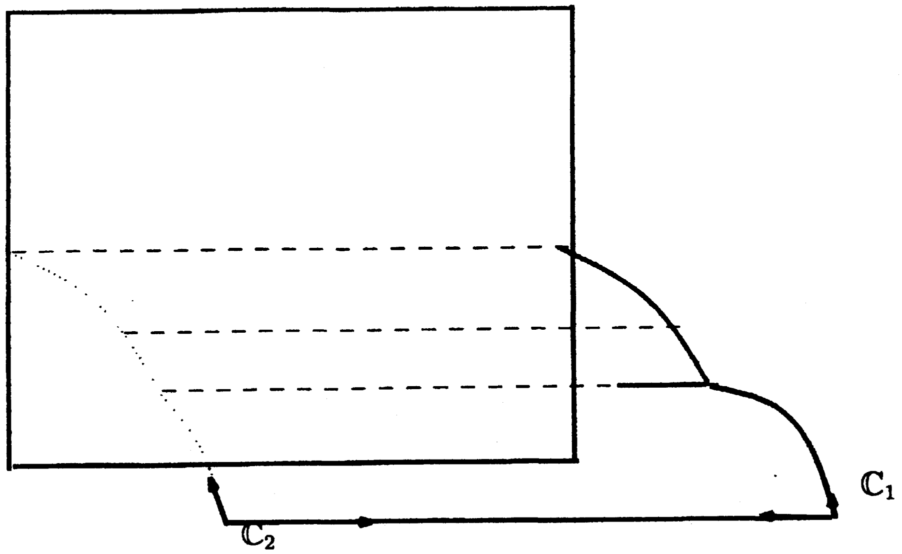

FIGURE 13

$$\begin{cases} l_7 &= l_6 \\ m_7 &= \frac{m_6+1-\alpha_3-\alpha_4}{n_6} \\ n_7 &= n_6 \end{cases} \tag{39}$$

In the coordinate, the equation is given by the following.

$$\begin{cases} \frac{dl_7}{dt} = tl_7 + 4l_7n_7 + (n_7)^2 + 4l_7m_7(n_7)^2 \\ \quad - 2\alpha_2(l_7)^2 + 2\alpha_2l_7n_2 + 2\alpha_2(l_7)^2m_7n_7 \\ \quad + (\alpha_2)^2(l_7)^2 + 4\alpha_3l_7n_7 + 2\alpha_2\alpha_3(l_7)^2 + 2\alpha_4l_7n_7 \\ \quad + \alpha_2\alpha_4(l_7)^2 \\ \frac{dm_7}{dt} = -1 - tm_7 + 4m_7n_7 - 3(m_7)^2(n_7)^2 + 2\alpha_3 - 4\alpha_3m_7n_7 \\ \quad - (\alpha_3)^2 + \alpha_4 - 2\alpha_4m_7n_7 - \alpha_3\alpha_4 \\ \frac{dn_7}{dt} = -1 + 2m_7(n_7)^3 - 2(n_7)^2 + tn_7 + 2\alpha_3(n_7)^2 + \alpha_4(n_7)^2 \end{cases} \tag{40}$$

$$\begin{cases} L_7 &= L_6 \\ M_7 &= \frac{M_6+1-\alpha_3-\alpha_4}{N_6} \\ N_7 &= N_6 \end{cases} \tag{41}$$

In the coordinate, the equation is given by the following.

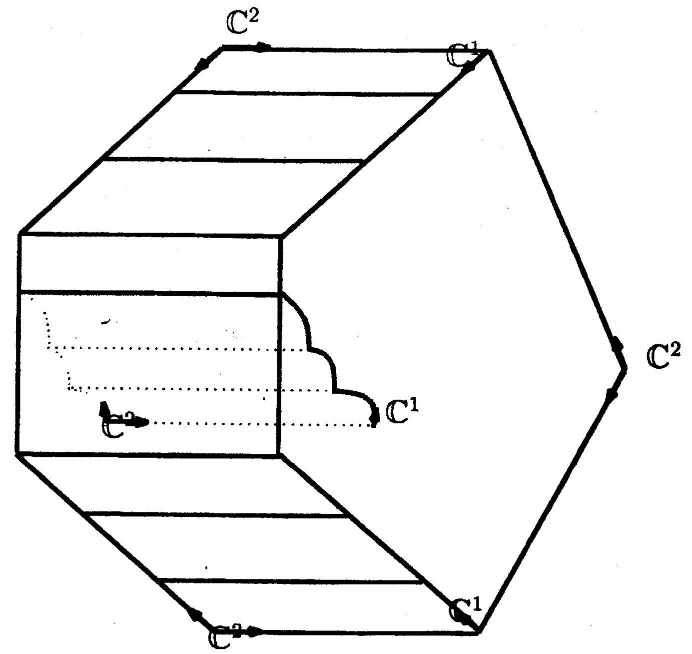

FIGURE 14

$$
\begin{cases}
\dfrac{dL_7}{dt} = -tL_7 + 4L_7N_7 - (L_7N_7)^2 - 4L_7M_7(N_7)^2 + 2\alpha_2 \\
\qquad - 2\alpha_2L_7N_7 - 2\alpha_2M_7N_7 - (\alpha_2)^2 - 4\alpha_3L_7N_7 \\
\qquad - 2\alpha_2\alpha_3 - 2\alpha_4L_7N_7 - \alpha_2\alpha_4 \\
\dfrac{dM_7}{dt} = -1 - tM_7 + 4M_7N_7 - 3(M_7N_7)^2 + 2\alpha_3 - 4\alpha_3M_7N_7 - (\alpha_3)^2 \\
\qquad + \alpha_4 - 2\alpha_4M_7N_7 - \alpha_3\alpha_4 \\
\dfrac{dN_7}{dt} = -1 + 2(M_7)^2 + tN_7 + 2M_7(N_7)^2 + 2\alpha_3(N_7)^2 + \alpha_4(N_7)^2
\end{cases}
\tag{42}
$$

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